Assessing Mathematical Reasoning: An Action Research Project
by Jill Thompson

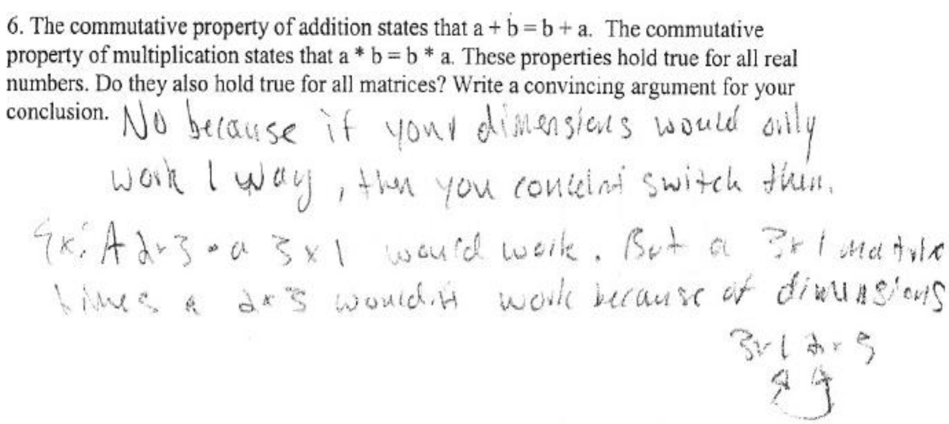
Excerpts from an IRB-Approved study conducted in 2006

Focus

Because reasoning is such a critical cognitive skill, it is important for teachers to know how to measure it. Teachers regularly use a variety of assessment practices in their classrooms, including observations, discussion, performance tasks, and more traditional assessments such as multiple-choice or short answer tests. Because formal assessment plays such an important role in assigning course grades or evaluating the effectiveness of instruction both within the classroom and increasingly within school districts, the purpose of this study is to seek an answer to this question: what can formal assessment tasks tell us about students' mathematical reasoning?

Findings (Excerpt)

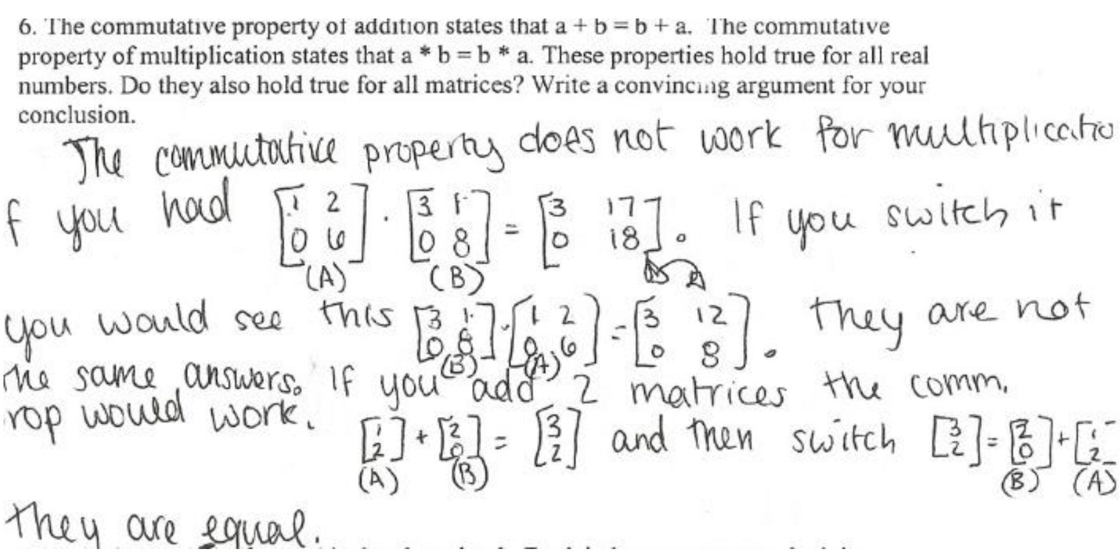
The most interesting question to analyze was question 6, where none of the students responded in a completely satisfactory manner. This question asked students to analyze matrix addition and multiplication to determine whether the commutative property might apply:


6. The commutative property of addition states that a + b = b + a. The commutative property of multiplication states that a * b = b * a. These properties hold true for all real numbers. Do they also hold true for all matrices? Write a convincing argument for your conclusion.

No because if your dimensions would only work 1 way, then you couldn't switch them. Ex: A 2×3 • a 3×1 would work. But a 3×1 matrix times a 2×3 wouldn't work because of dimensions 3×1 2×3

The example shown above represents a partially correct response, but is significant because the student's attempts to generalize. Only two students tried to make a general argument about multiplication using the dimensions of the matrices. While this response shows a high level of reasoning in the connection between the matrix multiplication algorithm and the commutative property, the response also indicates minimal understanding or use of a formal reasoning process involving a counterexample (27% of the students used the approach of finding a counterexample for multiplication).

Given that it is more difficult to offer proof that a statement is true, it is not surprising that none of the students were successful in providing a reasonable argument about addition. Only 15% of the students even attempted to address addition, and most did so using a specific example, as shown in the response below:


6. The commutative property of addition states that a + b = b + a. The commutative property of multiplication states that a * b = b * a. These properties hold true for all real numbers. Do they also hold true for all matrices? Write a convincing argument for your conclusion.

The commutative property does not work for multiplicatio f you had $\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}_{(A)} \cdot \begin{bmatrix} 3 & 1 \\ 0 & 8 \end{bmatrix}_{(B)} = \begin{bmatrix} 3 & 17 \\ 0 & 18 \end{bmatrix}$. If you switch it you would see this $\begin{bmatrix} 3 & 1 \\ 0 & 8 \end{bmatrix}_{(B)} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}_{(A)} = \begin{bmatrix} 3 & 12 \\ 0 & 8 \end{bmatrix}$. They are not the same answers. If you add 2 matrices the comm. rop would work. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(A)} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{(B)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and then switch $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{(B)} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(A)}$ they are equal.

It is important to note that, while students had previous experience with algebraic proofs in this class, they had not been asked to prove general arguments about quantities. Therefore, this task was completely novel, and drew primarily on the prior knowledge and reasoning capacity of the students rather than the accumulated effect of recent instruction. Some of the incorrect responses contained some very interesting connections, as in the cases where students tried to extend real number properties to matrices because matrices are composed of real numbers. Expanding on this reasoning could ultimately have led to a valid and defensible argument, and these students demonstrated a novel approach and sophisticated connection between ideas that was not evident in the responses of students who followed a more conventional path. It was on this problem that the scoring rubric was the least helpful in making a quantitative assessment of student reasoning.

Not only was this task informative about reasoning, it is also possible to make evaluations of students' procedural knowledge. In the example, close examination reveals that the student has problems arriving at the 2^{nd} row 2^{nd} column element each time she multiplies, possibly indicating a problem with the matrix multiplication algorithm itself. Every student's response provided a wealth of information that would not have been evident in tasks at lower levels of cognitive demand or multiple-choice format.

Conclusion and Implications (Excerpt)

A surprising result of this study highlights the importance of using assessment to evaluate and modify instructional practice. It was clear from the results of this study that the understanding of many students could be described as procedural. If students are instructed with a focus on mathematics as procedure, they are likely to rely on procedure in attempting to explain their approach to a task, as evidenced by common themes identified in the student work. It is clear that instruction must be modified to help students develop a view of mathematics as reasoning rather than procedure. To this end, problems such as the one dealing with the commutative property present a unique opportunity if they are used to help develop reasoning in the classroom community as a whole. All of the elements of a sound mathematical argument were present in the students' responses, yet students had no opportunity to share them with each other. Vygotsky's theory of social constructivism suggests that learning occurs when one shares ideas with more capable peers, lending support to the idea that group tasks are useful in helping students develop understanding (NCRMSE, 1991, p. 12) and therefore reasoning.

To return to the one issue that may have the most alarming implications, the difficulty in scoring the complex assessment items raises questions about whether standardized test items will ever be able to adequately measure reasoning. NCTM'S coherence standard argues for a balanced approach to assessment where the various types and phases of assessment are matched to their purpose (1995, p. 21). In order to measure the important standard of mathematical reasoning, students must be provided with instruction that allows them to invent, test and support their own ideas, and assessment items must measure whether students are able to apply their knowledge in novel situations (Battista, 1999, p. 15). Yet, with the unprecedented pressure of the accountability movement and massive revisions to state curriculum standards, there is evidence that instruction is increasingly focused on test preparation and that the quality of instruction is actually decreasing (Popham, 2004, p. 31). One can only hope that future studies confirm the value of focusing instruction on important standards such as reasoning, and that doing this will naturally lead to better performance on standardized tests.

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